Main theorem last class:

For a power series $\sum a_n x^n$ there exists $R \geq 0$ (possibly $R = \infty$) s.t.

- $\sum a_n x^n$ converges if $|x| < R$
- " diverges " $|x| > R$.

Essentially same result holds for power series of the form

$\sum a_n (x - x_0)^n$ x_0 fixed number

Example: $\sum_{n=0}^{\infty} \frac{1}{2^n n} (x-3)^n$

check: radius of converge $R = 2$!

formal way: let $y = x - 3$ $\rightarrow$ get power series $\sum \frac{1}{2^n n} y^n$

use theorem from last class

$\Rightarrow$ series $\sum \frac{1}{2^n n} y^n$ converges for $|y| < 2$

$\Rightarrow$ original series converges for $|x-3| < 2$

$\Leftrightarrow \quad -2 < x-3 < 2 \qquad | +3$

$\Leftrightarrow \quad 1 < x < 5$

Can define function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{for } |x| < R$$

$f(x)$ is the limit of the partial sums $\underbrace{\sum_{n=0}^{k} a_n x^n}_{\text{polynomial}}$

polynomials are continuous $\xrightarrow{?}$ is f also continuous?

⚠ Warning

A limit of continuous functions may **not** be continuous

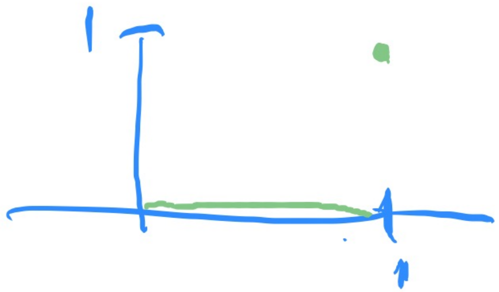
__Example:__ Let $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = x^n \quad \leftarrow \text{continuous}$$

check:
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } x<1 \\ 1 & \text{if } x=1 \end{cases}$$

But the function
$$f(x) = \begin{cases} 0 & x<1 \\ 1 & x=1 \end{cases}$$

is __not__ continuous at 1



Ch. 24 Uniform Convergence

There are different notions of convergence for functions!

Def 1 pointwise convergence:

$S \subset \mathbb{R}$

$f_n : S \to \mathbb{R}$ functions

$f : S \to \mathbb{R}$ a function

we say that $(f_n)_n$ converges pointwise to f

(notation $f_n \to f$ pointwise)

if $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in S$

have seen: $f_n(x) = x^n$ converges pointwise

to $f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$

here $S = [0,1]$.

Def. 2 uniform convergence

$S \subset \mathbb{R}$, $f_n, f : S \to \mathbb{R}$ functions

we say the functions f_n converge uniformly to f

if for every $\varepsilon > 0$ we can find an $N \in \mathbb{N}$

such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in S$

Remark: The point here is that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$

Examples

$S = \mathbb{R}$

① let $f_n(x) = \frac{1}{n} \sin nx$

we see $|f_n(x)| = |\frac{1}{n} \sin nx| \le \frac{1}{n}$

$\Rightarrow$ if $\varepsilon > 0$, pick an integer $N > \frac{1}{\varepsilon}$

$\Rightarrow$ if $n \ge N$ then

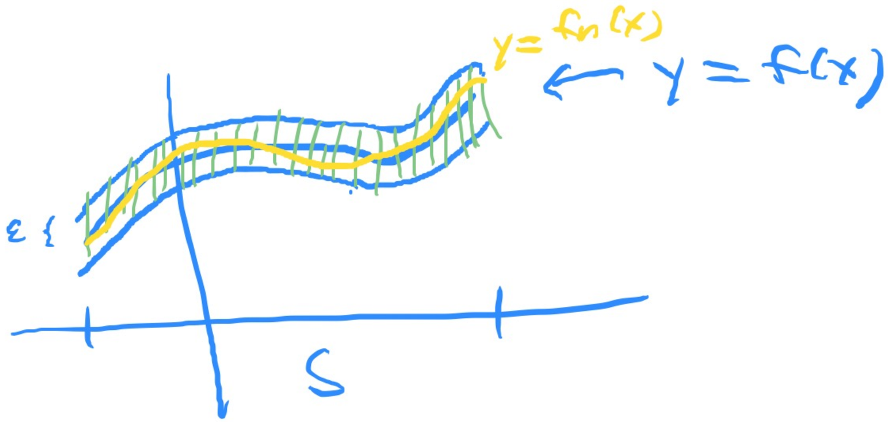
$$|f_n(x) - 0| = |\frac{1}{n} \sin nx| \le \frac{1}{n} \le \frac{1}{N} < \varepsilon$$

for all x in $\mathbb{R}$

Result: f_n converges uniformly to the zero function f

i.e. $f(x) = 0 \ \forall x$.

Illustration of uniform convergence.



if $|f_n(x) - f(x)| < \varepsilon$

for all x.

$\Longrightarrow$ the graph of f_n has to be between the functio $f(x)+\varepsilon$ and $f(x)-\varepsilon$

i.e. it has to be in the green region

⚠ If $f_n \to f$ pointwise

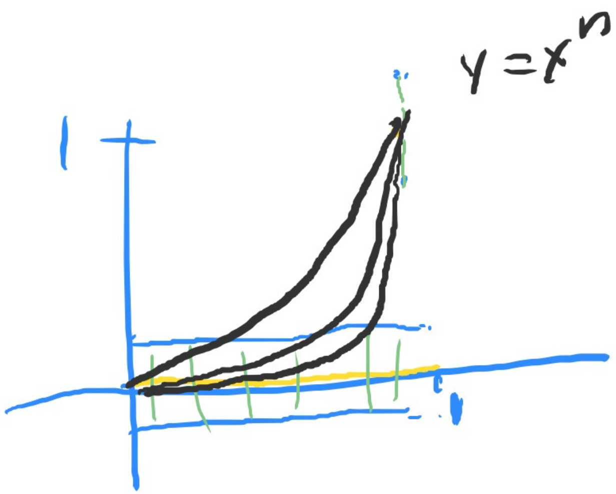
it may not necessarily converge uniformly

Example 2

$f_n(x) = x^n$, $S = [0,1]$

have seen $f_n(x) \to f$ pointwise

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$



y = f(x)

pick $\varepsilon = .02$

in order to show that $f_n \to f$ uniformly

we would need to find an f_n

whose graph is in the green region

$f_n(x) = x^n$

picture suggests: whatever n we pick

the graph of f_n will not be in green region formal proof.

want to show $f_n \not\Rightarrow f$ uniformly

i.e. need to find $\varepsilon > 0$

such that for every $n \in \mathbb{N}$ we can find x with

$$|f(x) - f_n(x)| > \varepsilon$$

let $\varepsilon = .2$

know: $f_n(x) = x^n$ is a cont. function.

$f_n(0) = 0$ $f_n(1) = 1$

$\Rightarrow$ $\exists$ x s.t. $f_n(x) = 1/2$ (i.e. $x = \sqrt[n]{1/2}$)

$x \neq 1$ obviously $\Rightarrow f(x) = 0$

$\Rightarrow |f_n(x) - f(x)| = |1/2 - 0| = 1/2 > .2$

$\Rightarrow$ no uniform convergence

have seen: · uniform convergence is stronger than pointwise convergence.

· pointwise limit of contin. functions may not be continuous.

next time we'll show:

uniform limit of cont. functions is again continuous